§3.2 Multiple fivebranes and non-Abelian MTCs

Consider now the (2,0)-theory of N M5 branes $(N>1)$ on $L(k,1)_b \times M_3$:

$$\begin{tikzcd}
 & \text{6d } (2,0)\text{-th on } L(k,1)_b \times M_3 \text{ reduce along} \arrow[dl, "M_3"'] \arrow[dr, "L(k,1)_b"] & \\
\text{3d } \mathcal{N}=2\text{-th } T_N[M_3] \text{ on } L(k,1)_b & & \text{complex CS-th on } M_3
\end{tikzcd}$$

The 3d-3d correspondence now reads

$$Z_{T_N[M_3]}\left[L(k,1)_b\right] = Z_{CS}^{(k,\sigma)}\left[M_3 ; GL(N,\mathbb{C})\right]$$

In the following we want to focus on "Seifert manifolds" atmitting a fibration structure :

$$\begin{tikzcd}
S^1 \arrow[r, hook] & M_3 \arrow[r, "\pi"] & \Sigma \\
 & & \text{Riemann-surface} \arrow[u]
\end{tikzcd}$$

Basic example $M_3 = S^1 \times \Sigma_1$

$\rightarrow$ geometry becomes

symmetries: $U(1)_N$ $SU(2)_R$

$\circlearrowright$ $\circlearrowright$

space-time : $L(k,1)_b \times T^*\Sigma \times S^1 \times \mathbb{R}^3$

$\cup$

N fivebranes: $L(k,1)_b \times \Sigma \times S^1$

$\rightarrow$ do topological twist along Riemann surface Σ

N M5's

$\downarrow S^1$

5d $\mathcal{N}=2$ SYM

$\downarrow \Sigma \subset T^*\Sigma$

3d $\mathcal{N}=4$ sigma model with target $\mathcal{M}_H(\Sigma; G)$ (Hitchin moduli space)

$$S = \int_M \sqrt{h}\, d^3x \Big(\frac{1}{2} g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial^\mu \varphi^j + \varepsilon_{IJ}\, \chi^I_\mu \nabla^\mu \bar{\eta}^J + \frac{1}{2}\frac{1}{\sqrt{h}} \varepsilon^{\mu\nu\rho} \varepsilon_{IJ}\, \chi^I_\mu \nabla_\nu \chi^J_\rho \Big)$$

where $\varphi^i : M \longrightarrow \mathcal{M}_H(\Sigma; G)$ are bosonic fields and χ^I_μ, η^J are their fermionic superpartners

$\rightarrow$ R-symmetry group enhanced to

$$SU(2)_R \times \underset{\overset{\cup}{U(1)_N \circlearrowright T^*\Sigma}}{SU(2)_N} \quad \text{(acts on fibers)}$$

We note that $\mathcal{M}_H(\Sigma; G_\mathbb{C})$ is the moduli space of G-Higgs bundles over Σ:

$$\mathcal{M}_H(\Sigma; G_\mathbb{C}) = \left\{ (A, \phi) \,\middle|\, \begin{array}{l} F_A - \phi \wedge \phi = 0 \\ d_A \phi = d_A^\dagger \phi = 0 \end{array} \right\} / \mathcal{G}$$

where $\phi \in \Omega^1(\Sigma, \mathfrak{g})$ is adjoint-valued one-form

$\longrightarrow$ weakly gauging $U(1)_\beta$ generated by $j_N^3 - j_R^3$ leads to 3d $\mathcal{N}=2$ theory with R-symmetry $U(1)_{R'}$ generated by $j_N^3 + j_R^3$

$U(1)_\beta$ acts on target $\mathcal{M}_H(\Sigma; G)$ as

$$U(1)_\beta : (A, \phi) \mapsto (A, e^{i\theta}\phi)$$

where each point in $\mathcal{M}_H(\Sigma; G)$ is represented by (A, ϕ)

$\rightarrow$ gaugin $U(1)_\beta$ gives rise to 3d $\mathcal{N}=2$ theory $T[\Sigma \times S^1; \beta]$ ("β-deformation," discuss later)

Question: What does the β-deformation means for complex CS-th on M_3 ?

For $M_3 = \Sigma \times S^1$ and $\beta = 1$, get:

$$Z_{CS}[\Sigma \times S^1; G_{\mathbb{C}}] = \dim \mathcal{H}_{CS}(\Sigma; G_{\mathbb{C}})$$

$\mathcal{H}_{CS}(\Sigma; G_{\mathbb{C}})$ is infinite-dim.

$\rightarrow$ β-def. regularizes it

To understand this, reduce 6d (2,0)-th on S^1-fiber of $L(k,1)$

→ 5d $\mathcal{N}=2$ SYM on $S^2 \times (\Sigma \times S^1)$

→ Lorentz and R-symmetry is broken to:

$$SO(5)_L \times SO(5)_R \longrightarrow \underset{\underset{S^2}{\circlearrowleft}}{SO(2)_L} \times \underset{\underset{\underset{\underset{U(1)_L \subset SO(3)_L}{\circlearrowright}}{\Sigma \times S^1}}{\circlearrowleft}}{SO(3)_L} \times U(1)_N \times SU(2)_R$$

top. twist → identify new Lorentz group $U(1)'$ with diagonal subgroup $U(1)_L \times U(1)_N$

the spectrum of 5d SYM transforms as follows:

5d	$SO(5)_L \times SO(5)_R$	field	$SO(2)_L$	$\times U(1)_L$	$\times U(1)_N$	$\times U(1)_R$
A^{5d}	$(5,1)$	A	0	± 2	0	0
		A_0	0	0	0	0
		B	± 2	0	0	0
ϕ^{5d}	$(1,5)$	ϕ	0	0	± 2	0
		ϕ_0	0	0	0	0
		Y	0	0	0	± 2
λ^{5d}	$(4,4)$	λ	± 1	± 1	± 1	± 1

top twist ⟶ ϕ becomes one-form on Σ

⟶ $\mathcal{A} = A + i\phi$ and $\mathcal{A}_0 = A_0 + i\phi_0$ become connection of complex CS-th along Σ and S^1

$U(1)_\beta$ acts as:

$$\theta \in U(1)_\beta : \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta\,\phi_1 - \sin\theta\,\phi_2 \\ \sin\theta \cdot \phi_1 + \cos\theta \cdot \phi_2 \end{pmatrix}$$

Alternatively, we can do a top. twist along $L(k,1)$:

$$\begin{tikzcd}
\text{N fivebranes:} & L(k,1)_b & \times & \Sigma & \times & S^1 \\
& \cap & & \cap & & \\
\text{spacetime:} & T^*L(k,1)_b & \times & T^*\Sigma & \times & S^1 \\
& \circlearrowright & & \circlearrowright & & \\
\text{symmetries:} & SU(2)_R & & U(1)_N & &
\end{tikzcd}$$

Know

$$\begin{tikzcd}
S^1 \arrow[r, hook, "\text{degree } k"] & L(k,1) \arrow[d] \\
& \mathbb{P}^1
\end{tikzcd}$$

⟶ take $\mathbb{R}^2_{-\beta}$ to be cotangent bundle of $\mathbb{P}^1$

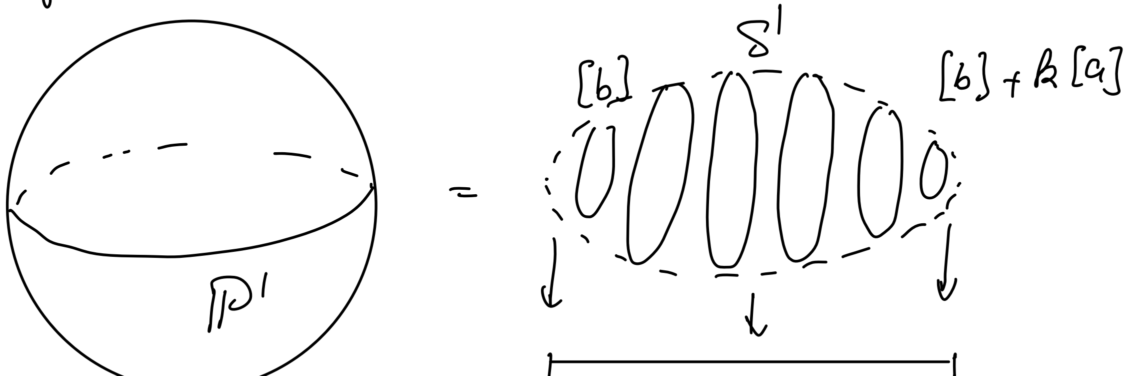
Thus we arrive at the correspondence:

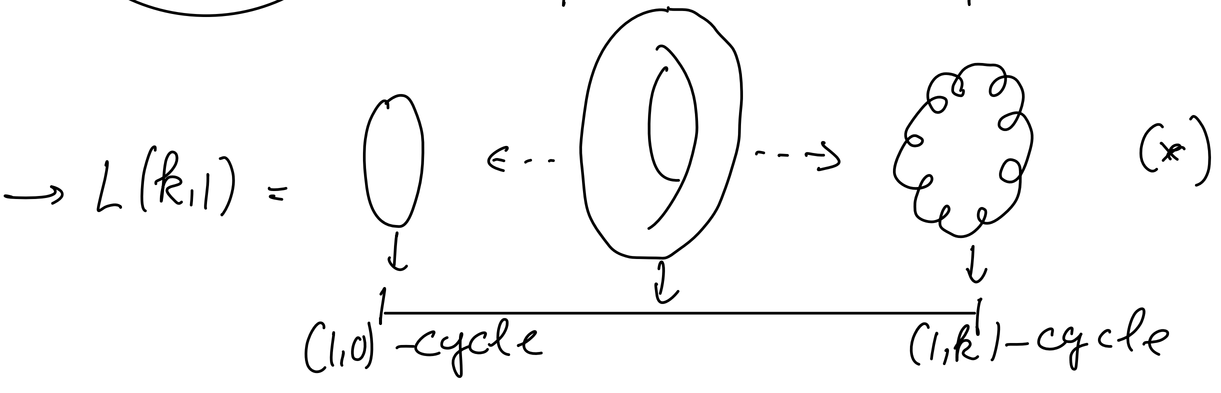
$$\dim_\beta \mathcal{H}_{CS}(\Sigma; G_{\mathbb{C}}) = Z_{CS}[\Sigma \times S^1; G_{\mathbb{C}}, \beta]$$
$$= Z^{\text{twisted}}_{T[L(k,1); \beta]}[\Sigma \times S^1]$$

Lens space theory $T[L(k,1)]$ from brane constructions

Regard $L(k,1)$ as T^2 fibration over interval

We have $H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ generated by $[a]$ and $[b]$





→ make following choice of spacetime coordinates:

spacetime	0	1	2	3	4	5	6	7	8	9	10
M5	–	–	–								
geometry	Σ		S^1	$\mathbb{R}^2_{\beta}$		$\mathbb{R}_{Hopf}$	Interval	$\mathbb{R}^2_{\beta}$		T^2	

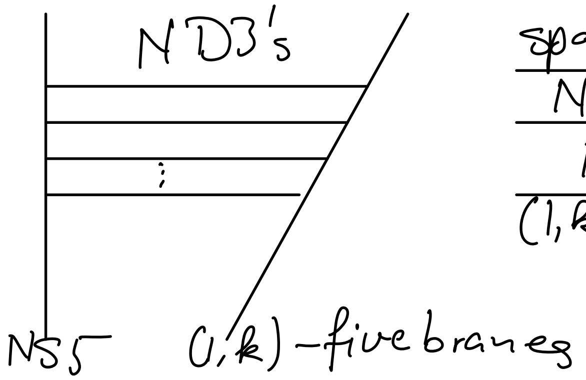
$\mathbb{R}^2_{\beta}$ (3,4) ↑ cotangent fiber of Σ

$\mathbb{R}^2_{\beta}$ (7,8) ↑ cotangent fiber of $\mathbb{P}^1$

Use duality

M-th on $T^2 \longleftrightarrow$ type IIB on S^1

→ M5-branes wrapping torus in (*) give rise to N D3-branes ending on a NS5-brane on one side of the interval and a (1,k)-brane on the other side:



space-time	0	1	2	3	4	5	6	7	8	9
N D3's	–	–	–	·	·	·	H	·	·	·
NS5	–	–	–	–	–	–	·	·	·	·
(1,k)-brane	–	–	–	–	–	\	·	·	·	\